A Unified Algebraic Approach to Few and Many-Body Hamiltonians having Linear Spectra

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Abstract

We develop an algebraic approach for finding the eigenfunctions of a large class of few and many-body Hamiltonians, in one and higher dimensions, having linear spectra. The method presented enables one to exactly map these interacting Hamiltonians to decoupled oscillators, thereby, giving a precise correspondence between the oscillator eigenspace and the wavefunctions of these quantum systems. The symmetries behind the degeneracy structure and the commuting constants of motion responsible for the quantum integrability of some of these models are made transparent. Apart from analysing a number of well-known dynamical systems like planar oscillators with commensurate frequencies, both with or without singular inverse-square terms and generalized Calogero-Sutherland type models, we also point out a host of other examples where the present approach can be profitably employed. We further study Hamiltonians having Laughlin wavefunction as the ground-state and establish their equivalence to free oscillators. This reveals the underlying linear $W_{1+\infty}$ symmetry algebra unambiguously and establishes the Laughlin wavefunction as the highest weight vector of this algebra.

1. INTRODUCTION

Few and many-body Hamiltonians having linear energy spectra appear in the description of various physical phenomena. A few well-known examples, sharing this common property, are electrons in a constant magnetic field [1], isotropic and anisotropic oscillators, with or without singular inverse-square terms [2–9] and Calogero-Sutherland (CS) type systems [10,11]. In recent times, the anisotropic oscillator has attracted considerable attention in the context of of the descriptions of the intrinsic states of rotating deformed nuclei [5,12], super-deformed high-spin states of spheroidal nuclear shapes [13] and fractional quantum Hall effect [14]. CS model and its generalizations have also been studied quite extensively in the current literature. They have found physical applications in various fields such as the universal conductance fluctuations in mesoscopic systems [15], quantum Hall effect [16], wave propagation in stratified fields [17], random matrix theory [11,18], fractional statistics [19], gravity [20] and gauge theories [21].

The linearity of the energy eigenvalues of these diverse interacting systems naturally suggests a probable connection with decoupled harmonic oscillators. This correspondence will, not only provide an elegant explanation for the linearity of the spectra, but also facilitate the characterization of the dynamical symmetries underlying these physical systems [22]. Recently, two of the present authors have shown that the many-body, Calogero-Sutherland model (CSM) of both A_{N-1} and B_N types can be mapped into a set of decoupled harmonic oscillators by means of a series of similarity transformations [23,24]. This mapping has led to a straightforward construction of the eigenfunctions, the commuting constants of motion, and enables one to write down the decoupled oscillator algebra for the *non-trivially* interacting CSM.

In this paper, we point out that a similar technique can be profitably employed for mapping a number of dynamical systems with linear spectra, to decoupled oscillators. As the first example, we establish the exact correspondence of the much studied, two-dimensional anisotropic oscillator with commensurate frequencies, both with or without singular inverse-

square terms, with the isotropic one. Making use of this connection, we *unambiguously* show that the symmetry algebra underlying the degeneracies of the anisotropic oscillator is a regular $SU(2)$ algebra. This result is subsequently extended to the N-dimensional case.

Section 3, deals with few-body Hamiltonians of CSM type. The algebraic structure, eigenfunctions and quantum integrability of a recently studied four particle model in one-dimension [25] is studied in detail. We then point out how the present method can be used to construct more general interacting models of CSM type.

In Section 4, we analyse various two-dimensional interacting Hamiltonins for electrons under the influence of both electromagnetic and Chern-Simons gauge fields. These models have Laughlin wavefunction as the ground-state and are also relevant for implementing the Jain picture of the fractional quantum Hall effect [26]. We explicitly show the connection of these interacting models with non-interacting harmonic oscillators. This reveals the underlying linear $W_{1+\infty}$ symmetry algebra with the Laughlin wavefunction as the highest weight vector of this algebra.

2. ANISOTROPIC OSCILLATORS

In the following, we show the precise correspondence of the two-dimensional anisotropic oscillator, both with or without singular inverse-square terms, to the isotropic one, establish the connection between their respective Hilbert spaces and write down the $SU(2)$ symmetry algebra underlying the degeneracies. There have been earlier attempts at explaining this degeneracy by $SU(2)$ Lie algebra; however, they were not completely satisfactory because of ambiguities in the definition of the generators [7,2,3]. This has led many authors to look for non-linear generalization of the $SU(2)$ algebra [6,8]. The mapping to isotropic oscillators naturally leads to an *unambiguous* definition of these generators. Later, this result is extended to the N-dimensional case, where the symmetry algebra is that of $SU(N)$.

The Hamiltonian of the two-dimensional anisotropic oscillator without singular inverse-square terms is given by ($\hbar = \omega = m = 1$)

$$H = -\frac{1}{2}(\partial_{x_1}^2 + \partial_{x_2}^2) + \frac{1}{2}(\frac{x_1^2}{\nu_1^2} + \frac{x_2^2}{\nu_2^2}) \qquad , \tag{1}$$

where, $\partial_{x_i} \equiv \frac{\partial}{\partial x_i}$ for $i = 1, 2$ and ν_1^{-1} and ν_2^{-1} are the dimensionless frequencies such that their ratio is a rational number. By a similarity transformation (ST), one gets

$$\tilde{H} \equiv \psi_0^{-1} H \psi_0 = \frac{1}{\nu_1} x_1 \partial_{x_1} + \frac{1}{\nu_2} x_2 \partial_{x_2} - \frac{1}{2} \partial_{x_1}^2 - \frac{1}{2} \partial_{x_2}^2 + E_0 \qquad , \tag{2}$$

where, $\psi_0 \equiv \exp\left(-\frac{1}{2\nu_1} x_1^2 - \frac{1}{2\nu_2} x_2^2\right)$ and $E_0 = \frac{1}{2\nu_1} + \frac{1}{2\nu_2}$. One more ST on $\tilde{H}$ yields

$$\bar{H} \equiv \hat{T}^{-1} \tilde{H} \hat{T} = \frac{1}{\nu_1} x_1 \partial_{x_1} + \frac{1}{\nu_2} x_2 \partial_{x_2} + E_0 \qquad , \tag{3}$$

here, $\hat{T} \equiv \exp\left(-\frac{\nu_1}{4} \partial_{x_1}^2 - \frac{\nu_2}{4} \partial_{x_2}^2\right)$.

By redefining $x_i = y_i^{\frac{1}{\nu_i}}$, (3) becomes,

$$\bar{H} = y_1 \partial_{y_1} + y_2 \partial_{y_2} + E_0 \qquad ; \tag{4}$$

$\bar{H}$ can be easily connected to the isotropic oscillator by one more ST:

$$H_o \equiv \hat{U} \bar{H} \hat{U}^{-1} = -\frac{1}{2} \partial_{y_1}^2 - \frac{1}{2} \partial_{y_2}^2 + \frac{1}{2} y_1^2 + \frac{1}{2} y_2^2 + E_0 - 1 \qquad , \tag{5}$$

where, $\hat{U} \equiv \exp(-\frac{1}{2} y_1^2 - \frac{1}{2} y_2^2) \exp(-\frac{1}{4} \partial_{y_1}^2 - \frac{1}{4} \partial_{y_2}^2)$.

From (5), one can define $a_{o,i} = \frac{1}{\sqrt{2}}(y_i + \partial_{y_i})$ and $a_{o,i}^\dagger = \frac{1}{\sqrt{2}}(y_i - \partial_{y_i})$:

$$H_o = \frac{1}{2} \{a_{o,1} \ , \ a_{o,1}^\dagger\} + \frac{1}{2} \{a_{o,2} \ , \ a_{o,2}^\dagger\} + E_0 - 1 \qquad ,$$

with $[a_{o,i} \ , \ a_{o,j}^\dagger] = \delta_{ij}$ and $[H_o \ , \ a_{o,i}(a_{o,i}^\dagger)] = -a_{o,i}(a_{o,i}^\dagger)$. By an inverse ST, one can formally write down the spectrum generating algebra for the original Hamiltonian in (1) as

$$H = \frac{1}{2} \{a_{y_1}^- \ , \ a_{y_1}^+\} + \frac{1}{2} \{a_{y_2}^- \ , \ a_{y_2}^+\} + E_0 - 1 \qquad , \tag{6}$$

with

$$[a_{y_i}^- \ , \ a_{y_j}^+] = \delta_{ij} \qquad ,$$

and

$$[H \ , \ a_{y_i}^\pm] = \pm \, a_{y_i}^\pm \qquad . \tag{7}$$

Here, $a_{y_i}^- = \hat{M} a_{o,i} \hat{M}^{-1}$, $a_{y_i}^+ = \hat{M} a_{o,i}^\dagger \hat{M}^{-1}$ and $\hat{M} \equiv \psi_0 \hat{T} \hat{U}^{-1}$.

The complete set of eigenfunctions can be obtained from (3) by using $a^-_{x_i} = (\psi_0 \hat{T})\partial_{x_i}(\psi_0 \hat{T})^{-1}$ and $a^+_{x_i} = (\psi_0 \hat{T}) x_i (\psi_0 \hat{T})^{-1}$. In this case, it can be verified that,

$$
\begin{aligned}
H &= \frac{1}{2\nu_1}\{a^-_{x_1}\ ,\ a^+_{x_1}\} + \frac{1}{2\nu_2}\{a^-_{x_2}\ ,\ a^+_{x_2}\} \qquad , \\
[H\ ,\ a^\pm_{x_i}] &= \pm\frac{1}{\nu_i}\ a^\pm_{x_i} \qquad .
\end{aligned}
\tag{8}
$$

The ground-state is obtained by $a^-_{x_i}\psi_0 = 0$ and the generic, unnormalized, excited states are

$$
\psi_{n_1,n_2} = (a^+_{x_1})^{n_1}(a^+_{x_2})^{n_2}\psi_0 \qquad , \tag{9}
$$

with energy eigenvalues

$$
E_{n_1,n_2} = \frac{n_1}{\nu_1} + \frac{n_2}{\nu_2} + E_0 \qquad . \tag{10}
$$

It is clear that degeneracy occurs only when $\frac{n_i}{\nu_i} = m_i; m_i = 0, 1, 2, \cdots$, in which case $E_{m_1,m_2} = m_1 + m_2 + E_0$. In order to identify the symmetry algebra underlying the degeneracy, it is convenient to work in the y-basis. It is easy to see that $a^-_{y_i}\psi_0 = 0$ and the degenerate excited states can be obtained by the repeated application of $a^+_{y_i}$ on ψ_0:

$$
\psi_{m_1,m_2} = (a^+_{y_1})^{m_1}(a^+_{y_2})^{m_2}\psi_0 \qquad , \tag{11}
$$

with energy eigenvalues

$$
E_{m_1,m_2} = m_1 + m_2 + E_0 \qquad . \tag{12}
$$

The fact that the degenerate states are generated by the action of $a^+_{y_i}$ on the ground-state and the Hamiltonian, in terms of $a^+_{y_i}$ and $a^-_{y_i}$ is precisely that of isotropic oscillators, makes the identification of the symmetry behind the degeneracy completely transparent. It is well known that the symmetry behind the planar isotropic oscillator is $SU(2)$. One can now *unambiguously* construct an $SU(2)$ algebra describing the degeneracy of the original anisotropic case:

$$
J_+ = a^+_{y_1}a^-_{y_2}\ , \quad J_- = a^+_{y_2}a^-_{y_1} \qquad , \tag{13}
$$

and

$$J_0 = \frac{1}{2}(a^+_{y_1} a^-_{y_1} - a^+_{y_2} a^-_{y_2}) \qquad , \tag{14}$$

such that

$$[J_0\ ,\ J_\pm] = \pm J_\pm \qquad ,$$
$$[J_+\ ,\ J_-] = 2J_0 \qquad . \tag{15}$$

At this moment, one is naturally led to the question of the construction of the $SU(2)$ generators in terms of $a^+_{x_i}$ and $a^-_{x_i}$. It is easy to see that the generators connecting the degenerate states can be written as

$$\tilde{J}_+ = (a^+_{x_1})^{\nu_1}(a^-_{x_2})^{\nu_2} \ \text{, and } \ \tilde{J}_- = (a^-_{x_1})^{\nu_1}(a^+_{x_2})^{\nu_2} \qquad .$$

However, these generators do not lead to $SU(2)$ algebra, since $[\tilde{J}_+\ ,\ \tilde{J}_-]$ is not linear in J_0 [6,8].

In order to identify the regular $SU(2)$ generators, we prescribe below a procedure which makes use of the method of canonical conjugate (CC) [27,28]. One can define the CC for $A_i \equiv (a^-_{x_i})^{\nu_i}$, for $i = 1, 2$, as

$$[A_i\ ,\ F^\dagger_j] = \delta_{ij} \qquad . \tag{16}$$

Here,

$$F^\dagger_i = \frac{1}{\nu_i} A^\dagger_i \frac{1}{A_i A^\dagger_i}(a^+_{x_i} a^-_{x_i} + \nu_i) \qquad , \tag{17}$$

and $A^\dagger_i \equiv (a^+_{x_i})^{\nu_i}$. Using the above result, one can define the $SU(2)$ generators,

$$\hat{J}^+ = F^\dagger_1 A_2 \qquad , \qquad \hat{J}^- = F^\dagger_2 A_1 \qquad , \tag{18}$$

$$\hat{J}^0 = \frac{1}{2}(F^\dagger_1 A_1 - F^\dagger_2 A_2) = \frac{1}{2}(A^\dagger_1 A_1 - A^\dagger_2 A_2) \tag{19}$$

such that

$$[\hat{J}_0\ ,\ \hat{J}_\pm] = \pm\hat{J}_\pm \qquad , \qquad [\hat{J}_+\ ,\ \hat{J}_-] = 2\hat{J}_0 \qquad . \tag{20}$$

It can be easily checked that these generators commute with the Hamiltonian (1) and act properly in the degenerate space.

We now proceed to analyse the Hamiltonian of the most general two-dimensional anisotropic oscillator with singular inverse-square terms:

$$H_s = -\frac{1}{2}(\partial_{x_1}^2 + \partial_{x_2}^2) + \frac{1}{2}(\frac{x_1^2}{\nu_1^2} + \frac{x_2^2}{\nu_2^2}) + \frac{g_1^2}{2}x_1^{-2} + \frac{g_2^2}{2}x_2^{-2} \qquad . \tag{21}$$

Performing a ST by the ground-state wavefunction, ψ_s, one gets

$$\tilde{H}_s \equiv \psi_s^{-1} H \psi_s = \frac{1}{\nu_1}x_1\partial_{x_1} + \frac{1}{\nu_2}x_2\partial_{x_2} - \hat{O}_1 - \hat{O}_2 + E_0 \qquad , \tag{22}$$

where, $\psi_s \equiv x_1^{\alpha_1}x_2^{\alpha_2}\exp(-\frac{1}{2\nu_1}x_1^2 - \frac{1}{2\nu_2}x_2^2)$, $\alpha_i = \frac{1}{2}(1+\sqrt{1+4g_i^2})$, $\hat{O}_i \equiv \frac{1}{2}\partial_{x_i}^2 + \frac{\alpha_i}{x_i}\partial_{x_i}$ and $E_0 \equiv \frac{1}{2\nu_1} + \frac{1}{2\nu_2} + \frac{\alpha_1}{\nu_1} + \frac{\alpha_2}{\nu_2}$. One more ST on $\tilde{H}$ yields

$$\bar{H}_s \equiv \hat{T}_s^{-1}\tilde{H}\hat{T}_s = \frac{1}{\nu_1}x_1\partial_{x_1} + \frac{1}{\nu_2}x_2\partial_{x_2} + E_0 \qquad , \tag{23}$$

here, $\hat{T}_s \equiv \exp(-\frac{\nu_1}{2}\hat{O}_1 - \frac{\nu_2}{2}\hat{O}_2)$.

By redefining $x_i = y_i^{\frac{1}{2\nu_i}}$, (23) becomes,

$$\bar{H}_s = 2(y_1\partial_{y_1} + y_2\partial_{y_2}) + E_0 \qquad ; \tag{24}$$

$\bar{H}_s$ can be easily connected to the isotropic oscillator on a plane by one more ST:

$$H_{\rm iso} \equiv \hat{U}_s\bar{H}_s\hat{U}_s^{-1} = -(\partial_{y_1}^2 + \partial_{y_2}^2) + y_1^2 + y_2^2 + E_0 - 2 \qquad , \tag{25}$$

where, $\hat{U}_s \equiv \exp(-\frac{1}{2}y_1^2 - \frac{1}{2}y_2^2)\exp(-\frac{1}{4}\partial_{y_1}^2 - \frac{1}{4}\partial_{y_2}^2)$.

It is worth pointing out that the choice of the y_i variables ensures normalizability of the wavefunctions for H_s when they are constructed from those of the $H_{\rm iso}$.

In order to get the spectrum generating and degeneracy algebras, one can proceed in parallel to the anisotropic oscillator without singular terms. Defining $b_{x_i}^-[b_{y_i}^-] = (\psi_s\hat{T}_s)\partial_{x_i}[\partial_{y_i}](\psi_s\hat{T}_s)^{-1}$ and $b_{x_i}^+[b_{y_i}^+] = (\psi_s\hat{T}_s)x_i[y_i](\psi_s\hat{T}_s)^{-1}$, it is easy to see that

$$[b^-_{x_i}\ ,\ b^+_{x_j}] = [b^-_{y_i}\ ,\ b^+_{y_i}] = \delta_{ij} \qquad , \tag{26}$$

and

$$[H\ ,\ b^\pm_{x_i}(b^\pm_{y_i})] = \pm\frac{1}{\nu_i}\ b^\pm_{x_i}(\pm b^\pm_{y_i}) \qquad . \tag{27}$$

The ground-state wavefunction of the Hamiltonian H_s can be gotten by $b^-_{x_i}\psi_0 = b^-_{y_i}\psi_s = 0$ and all the excited states can be obtained by the repeated application of $b^+_{x_i}$ on ψ_s. The generic, unnormalized, excited state can be written as

$$\Psi_{n_1,n_2} = (b^+_{x_1})^{2n_1}(b^+_{x_2})^{2n_2}\psi_s \qquad , \tag{28}$$

and the corresponding energy eigenvalues are

$$E_{n_1,n_2} = 2(\frac{n_1}{\nu_1} + \frac{n_2}{\nu_2}) + E_0 \qquad . \tag{29}$$

As has been pointed out earlier, for the analysis of the degeneracy structure, it is advantageous to work with the $b^-_{y_i}$ and $b^+_{y_i}$ operators. The generic excited state obtained by the repeated action of $b^+_{y_i}$ are

$$\psi_{m_1,m_2} = (b^+_{y_1})^{m_1}(b^+_{y_2})^{m_2}\psi_s \qquad , \tag{30}$$

with energy eigenvalues

$$E_{m_1,m_2} = 2(m_1 + m_2) + E_0 \qquad . \tag{31}$$

It is clear from (29) and (31) and also from (26) and (27) that $b^+_{y_i}$ creates states whose eigenvalues are ν_i times the eigenvalues of the states created by $b^+_{x_i}$.

Using $b^\pm_{y_i}$, one can construct the generators of $SU(2)$ algebra in a manner analogous to the previously treated example.

It is easy to see that the following generators

$$\tilde{J}_+ = (b^+_{x_1})^{2\nu_1}(b^-_{x_2})^{2\nu_2}\ \text{, and }\ \tilde{J}_- = (b^-_{x_1})^{2\nu_1}(b^+_{x_2})^{2\nu_2} \qquad ,$$

do not belong to the $SU(2)$ algebra, since $[\tilde{J}_+\ ,\ \tilde{J}_-]$ is not linear in J_0 [6,8].

Akin to the anisotropic oscillator without singular terms, the regular $SU(2)$ generators can be constructed by the method of CC [27,28]. One can define the CC for $B_i \equiv (b^-_{x_i})^{2\nu_i}$, for $i = 1, 2$, as

$$[B_i \ , \ G^\dagger_j] = \delta_{ij} \tag{32}$$

Here,

$$G^\dagger_i = \frac{1}{2\nu_i} B^\dagger_i \frac{1}{B_i B^\dagger_i} (b^+_{x_i} b^-_{x_i} + 2\nu_i) \qquad , \tag{33}$$

and $B^\dagger_i \equiv (b^+_{x_i})^{2\nu_i}$. Using these B_i and $G^\dagger_i$, we can define the $SU(2)$ generators in exactly the same way as that of the earlier example.

Here, we would like to remark that, when $\nu_1 = \nu_2 = 1$, $x_1 \to \frac{1}{\sqrt{2}}(x_1 - x_2)$ and $x_2 \to \frac{1}{\sqrt{2}}(x_1 + x_2)$, the model in (21) goes to a special case of the two particle Calogero model of B_N type [24]. This can be further reduced to the two particle Calogero model of A_{N-1} type [23] by choosing $g_2 = 0$. Hence, our analyses are also applicable to the above CS type models; one needs only to symmetrize the wavefunctions appropriately.

The above analysis for the two-dimensional anisotropic oscillator can be immediately generalized to the N dimensional case. The Hamiltonian with singular terms reads as

$$H = -\frac{1}{2}\sum_{i=1}^{N} \partial^2_{x_i} + \frac{1}{2}\sum_{i=1}^{N} \frac{x_i^2}{\nu_i^2} + \sum_{i=1}^{N} \frac{g_i^2}{2} x_i^{-2} \qquad . \tag{34}$$

The following ST by $\hat{S} \equiv \prod_{i=1}^{N} x_i^{\alpha_i} \exp\left(-\frac{1}{2}\sum_{i=1}^{N} \frac{x_i^2}{\nu_i}\right) \exp(-\sum_{i=1}^{N} \frac{1}{2}\nu_i \hat{P}_i)$; where, $\hat{P}_i \equiv \frac{1}{2}\partial^2_{x_i} + \frac{\alpha_i}{x_i}\partial_{x_i}$, yields

$$\bar{H} \equiv \hat{S}^{-1} \tilde{H} \hat{S} = \sum_{i=1}^{N} \frac{1}{\nu_i} x_i \partial_{x_i} + E_0 \qquad , \tag{35}$$

where, $\alpha_i = \frac{1}{2}(1 + \sqrt{1 + 4g_i^2})$ and $E_0 \equiv \sum_{i=1}^{N} (\frac{1}{2\nu_i} + \frac{\alpha_i}{\nu_i})$. Akin to the two-dimensional case, one can redefine $x_i = y_i^{\frac{1}{2\nu_i}}$: (35) becomes,

$$\bar{H} = 2\sum_{i=1}^{N} y_i \partial_{y_i} + E_0 \qquad . \tag{36}$$

Now, (36) can be shown to be equivalent to an isotropic harmonic oscillator in N dimension, with equal frequencies by the following ST,

$$H_o \equiv \hat{U}\bar{H}\hat{U}^{-1} = -\sum_{i=1}^{N} \partial_{y_i}^2 + \sum_{i=1}^{N} y_i^2 + E_0 - N \qquad , \tag{37}$$

where, $\hat{U} \equiv \exp(-\frac{1}{2}\sum_i y_i^2)\exp(-\frac{1}{4}\sum_i \partial_{y_i}^2)$.

A procedure similar to the two-dimensional case can be straightforwardly employed to show that the energy level degeneracy of the anisotropic oscillator in N dimensions is described by an $SU(N)$ algebra. The generators, analogous to the Fradkin tensor [29] for the isotropic case, are given by

$$F_{ij} = c_{y_i}^+ c_{y_j}^- \qquad ,$$

where, $c_{y_i}^- = \hat{S}\partial_{y_i}\hat{S}^{-1}$ and $c_{y_i}^+ = \hat{S}y_i\hat{S}^{-1}$. Another basis for the Fradkin tensor can be given by the method of canonical conjugate as

$$F'_{ij} = K_i^\dagger C_j \qquad ,$$

where,

$$C_i \equiv (c_{x_i}^-)^{2\nu_i} \;\; , \;\; K_i^\dagger = \frac{1}{2\nu_i} C_i^\dagger \frac{1}{C_i C_i^\dagger}(c_{x_i}^+ c_{x_i}^- + 2\nu_i) \qquad .$$

and

$$[C_i \;,\; K_j^\dagger] = \delta_{ij} \qquad .$$

Here, $C_i^\dagger \equiv (c_{x_i}^+)^{2\nu_i}$, $c_{x_i}^- = \hat{S}\partial_{x_i}\hat{S}^{-1}$ and $c_{x_i}^+ = \hat{S}x_i\hat{S}^{-1}$.

The above analysis can be easily applied to the anisotropic oscillator with commensurate frequencies and without singular inverse-square terms.

3. MANY-BODY INTERACTING MODELS

In the following, we first analyse a recently studied one-dimensional model of four identical particles with both two-body and four-body inverse-square interactions given by the Hamiltonian [25],

$$H = -\frac{1}{2}\sum_{i=1}^{4}\partial_{x_i}^2 + \frac{1}{2}\sum_{i=1}^{4}x_i^2 + g_1\sum_{\substack{i,j\\ i\neq j}}(x_i - x_j)^{-2} + g_2\sum_{\substack{\text{3independent}\\ \text{terms}}}(x_i + x_j - x_k - x_l)^{-2} \quad . \tag{38}$$

The correlated bosonic ground-state of H is given by

$$\psi_0 = \prod_{i<j}|x_i - x_j|^{\alpha} \prod_{\text{3indep.terms}}(x_i + x_j - x_k - x_l)^{\beta} \;\exp\{-\frac{1}{2}\sum_i x_i^2\} \quad ,$$

where, $\alpha = \frac{1}{2}(1+\sqrt{1+4g_1})$ and $\beta = \frac{1}{2}(-1+\sqrt{1+2g_2})$. By performing a ST on H with respect to ψ_0, one gets

$$\psi_0^{-1}H\psi_0 \equiv \tilde{H} = \sum_i x_i\partial_{x_i} - \hat{A} + E_0 \quad , \tag{39}$$

where,

$$\hat{A} = \frac{1}{2}\sum_i \partial_{x_i}^2 + \alpha\sum_{i\neq j}\frac{1}{x_i - x_j}\partial_{x_i} + \beta\sum_{\text{3indep.terms}}\frac{1}{(x_i + x_j - x_k - x_l)}(\partial_{x_i} + \partial_{x_j} - \partial_{x_k} - \partial_{x_l})$$

and $E_0 = 2 + 6\alpha + 3\beta$. Another ST by $\hat{S} = \exp\{-\hat{A}/2\}$ on $\tilde{H}$ diagonalizes it completely:

$$\hat{S}^{-1}\tilde{H}\hat{S} \equiv \bar{H} = \sum_i x_i\partial_{x_i} + E_0 \quad . \tag{40}$$

The explicit connection of H with the decoupled oscillators can be obtained by one more ST on $\bar{H}$

$$\hat{T}^{-1}\bar{H}\hat{T} = -\frac{1}{2}\sum_{i=1}^{4}\partial_{x_i}^2 + \frac{1}{2}\sum_{i=1}^{4}x_i^2 + E_0 - 2 \quad , \tag{41}$$

where, $\hat{T} \equiv \exp(-\frac{1}{2}\sum_{i=1}^{4}x_i^2)\exp(-\frac{1}{4}\sum_{i=1}^{4}\partial_{x_i}^2)$.

Akin to the anisotropic oscillator case, one can find the creation and annihilation operators for the above four-particle model:

$$H = \sum_{i=1}^{4}H_i + E_0 - 2 = \frac{1}{2}\sum_{i=1}^{4}\{a_i^- \;,\; a_i^+\} + E_0 - 2 \quad , \tag{42}$$

where, $H_i = \frac{1}{2}\{a_i^- \ , \ a_i^+\}$, $a_i^- = \hat{M} b_i \hat{M}^{-1}$, $a_i^+ = \hat{M} b_i^\dagger \hat{M}^{-1}$, $b_i = \frac{1}{\sqrt{2}}(x_i + \partial_{x_i})$, $b_i^\dagger = \frac{1}{\sqrt{2}}(x_i - \partial_{x_i})$ and $\hat{M} \equiv \psi_0 \hat{S} \hat{T}^{-1}$. It follows that

$$[a_i^- \ , \ a_j^+] = \delta_{ij} \qquad ,$$

and

$$[H_i \ , \ a_i^-(a_i^+)] = [H \ , \ a_i^-(a_i^+)] = -a_i^-(a_i^+) \qquad .$$

For the construction of the eigenfunctions of this model, one can also make use of (40) since, x_i and ∂_{x_i} serve as the creation and annihilation operators respectively. The ground-state can be chosen: $\partial_{x_i}\phi_0 = 0$, for $i = 1, 2, 3, 4$. The excited states are given by the monomials $\prod_i^4 x_i^{n_i}$; $n_i = 0, 1, 2, \cdots$. It is worth mentioning that for the normalizability of the wavefunctions, one needs to check that the action of $\exp\{-\hat{A}/2\}$ on an appropriate symmetric combination of the eigenstates of $\sum_{i=1}^4 x_i \partial_{x_i}$ yields a polynomial solution [23]. In the following, we present explicit forms of some eigenstates.

Case I: Eigenstates corresponding to the center-of-mass degree of freedom:

$$\psi_{n_1,0,0,0} = \psi_0 \ \exp\{-\frac{1}{2}\hat{A}\} R^{n_1} = \exp\{-\frac{1}{4}\sum_{N=1}^{4}\partial_{x_i}^2\} \ R^{n_1} \qquad , \tag{43}$$

where, $R = 4^{-1}(x_1 + x_2 + x_3 + x_4)$. This can be cast in the form [30],

$$\psi_{n_1,0,0,0} = 8^{-n_1} n_1! \psi_0 \sum_{\sum_{i=1}^4 m_i = n_1} \prod_{i=1}^{4} \frac{H_{m_i}(x_i)}{m_i!} \qquad . \tag{44}$$

Here, $H_{m_i}(x_i)$ are the Hermite polynomials.

Case II: Eigenstates corresponding to the radial degree of freedom [31]:

$$\psi_{0,n_2,0,0} = \psi_0 \ \exp\{-\frac{1}{2}\hat{A}\}(r^2)^{n_2} \qquad . \tag{45}$$

where, $r^2 \equiv \frac{1}{4}\sum_{\substack{i,j=1 \\ i<j}}^4 (x_i - x_j)^2$. It is easy to check that $\hat{A}(r^2)^n = 2n(n + \frac{1}{2} + 6\alpha + 3\beta)(r^2)^{n-1}$; using this, one gets

$$\begin{aligned} \psi_{0,n_2,0,0} &= \psi_0 \ (-1)^{n_2} \ n_2! \ \sum_{l=0}^{n_2} \frac{(-1)^{n_2}}{l!(n_2 - l)!} \frac{(n_2 + \frac{1}{2} + 6\alpha + 3\beta)!}{(l + \frac{1}{2} + 6\alpha + 3\beta)!} (r^2)^l \\ &= \psi_0 \ (-1)^{n_2} \ n_2! \ L_{n_2}^{\frac{1}{2}+6\alpha+3\beta}(r^2) \qquad . \end{aligned} \tag{46}$$

Here, $L_{n_2}^{\frac{1}{2}+6\alpha+3\beta}(r^2)$ is the Lagurre polynomial. In a similar way, one can find any generic excited state wavefunction.

One can also define $<< 0|S_n(\{a_i^-\}) =<< n|$ and $S_n(\{a_i^+\})|0 >= |n >$ as the bra and ket vectors; S_n is a symmetric and homogeneous function of degree n and $<< 0|a_i^+ = a_i^-|0 >= 0$. Since the oscillators are decoupled, the inner product between these bra and ket vectors proves that any ket $|n >$, with a given partition of n, is orthogonal to all the bra vectors, with different n and also to those with different partitions of the same n.

The quantum integrability of this model can be proved easily by noting that the set $\{H_1, H_2, H_3, H_4\}$ provides the four mutually commuting conserved quantities, i.e., $[H\ ,\ H_k] = [H_i\ ,\ H_j] = 0$. From this set, one can construct four linearly independent symmetric conserved quantities.

More general interacting models of the CSM type, which can be mapped to decoupled oscillators, can be constructed in the following manner.

One starts with the general Hamiltonian of the type

$$H = -\frac{1}{2}\sum_{i=1}^{N}\partial_{x_i}^2 + V(x_1, x_2, x_3, \cdots, x_N) \qquad , \tag{47}$$

having Φ_0 as the ground-state wavefunction and $V(x_i, x_2, x_3, \cdots, x_N) = \frac{1}{2\Phi_0}\sum_{i=1}^{N}\partial_{x_i}^2\Phi_0$. In order to bring H to the following form

$$\Phi_0^{-1}H\Phi_0 \equiv \tilde{H} = \sum_{i}^{N} x_i\partial_{x_i} - \hat{A} + E_0 \qquad , \tag{48}$$

the ground-state wavefunction must be of the form $\Phi_0 = GJ$; where, $G \equiv \exp\{-\frac{1}{2}\sum_{i=1}^{N} x_i^2\}$, $\hat{A} \equiv \frac{1}{2}\sum_{i=1}^{N}\partial_{x_i}^2 + \frac{1}{J}\sum_{i=1}^{N}\partial_{x_i}(\ln J)\partial_{x_i}$ and E_0 is the ground-state energy.

$\tilde{H}$ can be mapped to the Euler operator by another ST

$$\hat{S}^{-1}\tilde{H}\hat{S} \equiv \bar{H} = \sum_{i=1}^{N} x_i\partial_{x_i} + E_0 \qquad , \tag{49}$$

provided, the following equation holds,

$$\left[\tilde{H}, \exp\{-\hat{A}/2\}\right] = \left[\sum_i x_i\partial_{x_i}, \exp\{-\hat{A}/2\}\right] = \hat{A}\exp\{-\hat{A}/2\} \qquad . \tag{50}$$

The above condition restricts J to be a homogeneous function of the particle coordinates.

Now, it is easy to see that, the Hamiltonian in (47) can be mapped to free oscillators by a series of STs,

$$G\hat{E}\exp\{\hat{A}/2\}\Phi_0{}^{-1}H\Phi_0\exp\{-\hat{A}/2\}\hat{E}^{-1}G^{-1} = -\frac{1}{2}\sum_i \partial_{x_i}^2 + \frac{1}{2}\sum_i x_i^2 + (E_0 - \frac{1}{2}N) \qquad , \quad (51)$$

where, $\hat{E} \equiv \exp\{-\frac{1}{4}\sum_{i=1}^N \partial_{x_i}^2\}$.

However, it is important to check that, the action of $\exp\{-\hat{A}/2\}$ on an appropriate linear combination of the eigenstates of $\sum_{i=1}^N x_i\partial_{x_i}$ yields a polynomial solution. Otherwise, the resulting functions are not normalizable with respect to Φ_0^2 as a weight function. Appropriate choices of J will yield new solvable models having linear spectra. This case can also be generalized to the higher-dimensional interacting models [23,32,33].

4. LAUGHLIN WAVEFUNCTION AND DECOUPLED HARMONIC OSCILLATORS

In this section, we study some planar many-body Hamiltonians relevant for the description of the quantum Hall effect. These Hamiltonians describe electrons in a magnetic field, with two-body and three-body inverse-square interactions arising due to Chern-Simons gauge field and have Laughlin wavefunction [34] as the ground-state [35]. We explicitly prove that, this model can be exactly mapped to a set of free harmonic oscillators on the plane. As a consequence, the existence of a *linear* $W_{1+\infty}$ algebra with Laughlin wavefunction as its highest weight vector, is pointed out in a rather elegant and straightforward manner.

The relevant Hamiltonian is given by

$$\begin{aligned} H = \quad & \frac{1}{2}\sum_{i=1}^N(-4\partial_{\bar{z}_i}\partial_{z_i} + z_i\partial_{z_i} - \bar{z}_i\partial_{\bar{z}_i} + \frac{1}{4}\bar{z}_iz_i) + 2\eta\sum_{\substack{i=1\\ i\neq j}}^N \frac{1}{(z_i - z_j)}(\partial_{\bar{z}_i} - \frac{1}{4}z_i) \\ & - 2\eta\sum_{\substack{i=1\\ i\neq j}}^N \frac{1}{(\bar{z}_i - \bar{z}_j)}(\partial_{z_i} - \frac{1}{4}\bar{z}_i) + 2\eta^2 \sum_{\substack{i,j,k=1\\ i\neq j; i\neq k}}^N \frac{1}{(z_i - z_j)(\bar{z}_i - \bar{z}_k)} \qquad , \end{aligned} \quad (52)$$

where, $z_i = x_i + iy_i$. The ground-state of this model was found to be of Laughlin form,

$$\psi_0 = \prod_{i<j}(z_i - z_j)^\eta \ \exp\{-\frac{1}{4}\sum_i \bar{z}_iz_i\} \qquad .$$

By performing a ST, one gets

$$\psi_0^{-1} H \psi_0 \equiv \tilde{H} = \sum_i z_i \partial_{z_i} - \hat{A} + \frac{1}{2} N \qquad , \tag{53}$$

where, $\hat{A} \equiv 2\sum_i \partial_{\bar{z}_i}\partial_{z_i} + 2\eta \sum_{i\neq j}(\frac{1}{\bar{z}_i - \bar{z}_j}\partial_{z_i} - \frac{1}{z_i - z_j}\partial_{\bar{z}_i})$. It is easy to check that

$$[\sum_i z_i \partial_{z_i} \ , \ \hat{A}] = -\hat{A} + 4\pi\eta \sum_{\substack{i,j \\ i\neq j}} (z_i - z_j)\delta^2(z_i - z_j)\partial_{z_i} \qquad . \tag{54}$$

In the view of the δ-function identity *i.e.*, $x\delta(x) = 0$, the above equation reduces to

$$[\sum_i z_i \partial_{z_i} \ , \ \hat{A}] = -\hat{A} \qquad . \tag{55}$$

Performing a ST by $\exp\{-\hat{A}\}$, (53) becomes

$$\exp\{\hat{A}\} \ \tilde{H} \ \exp\{-\hat{A}\} \equiv \bar{H} = \sum_i z_i \partial_{z_i} + \frac{1}{2} N \qquad . \tag{56}$$

Finally, the following ST by $\hat{W} \equiv \exp\{2\sum_i \partial_{\bar{z}_i}\partial_{z_i}\}\exp\{\frac{1}{4}\sum_i \bar{z}_i z_i\}$ brings the above Hamiltonian to a Hamiltonian of N free harmonic oscillators,

$$\hat{W}^{-1} H \hat{W} = \frac{1}{2}\sum_{i=1}^{N}(-4\partial_{\bar{z}_i}\partial_{z_i} + z_i\partial_{z_i} - \bar{z}_i\partial_{\bar{z}_i} + \frac{1}{4}\bar{z}_i z_i) \qquad . \tag{57}$$

By defining $a_i^+ = \hat{S}^{-1} z_i \hat{S}$ and $a_i^- = \hat{S}^{-1}\partial_{z_i}\hat{S}$; where $\hat{S} = \psi_0 \exp\{\hat{A}\}$, and making use of (56), one can rewrite (52) as

$$H = \sum_i H_i + \frac{1}{2} N = \sum_i a_i^+ a_i^- + \frac{1}{2} N \tag{58}$$

where, $H_i \equiv a_i^+ a_i^-$, such that

$$[a_i^- \ , \ a_j^+] = \delta_{ij} \qquad ,$$

and

$$[H_i \ , \ a_j^\pm] = \pm a_j^\pm \delta_{ij} \qquad .$$

These N quantities H_i serve as the conserved quantities which are in involution, *i.e.*, $[H_i \ , \ H_j] = 0$.

Since a_i^- and a_i^+ obey non-interacting oscillator algebra, one can make use of this fact to define a *linear* $W_{1+\infty}$ algebra. The generators of the $W_{1+\infty}$ algebra, $L_{m,n} = \sum_{i=1}^N (a_i^+)^{m+1}(a_i^-)^{n+1}$, for $m, n \geq -1$ obey the linear relation [36]

$$[L_{m,n}\ ,\ L_{r,s}] = \sum_{p=0}^{\mathrm{Min}(n,r)} \frac{(n+1)!(r+1)!}{(n-p)!(r-p)!(p+1)!} L_{m+r-p,n+s-p} - (n \leftrightarrow s, m \leftrightarrow r) \quad . \quad (59)$$

The highest weight vector obtained from $L_{m,n}\psi_0 = 0$ for $n > m \geq -1$ is nothing but the Laughlin wavefunction.

In the following, we present two more two-dimensional models which can also be made equivalent to a set of decoupled oscillators. The first one is due to Rajaraman and Sondhi (RS) [37]

$$H_{RS} = \frac{1}{2}\sum_{i=1}^N(-4\partial_{\bar{z}_i}\partial_{z_i} + z_i\partial_{z_i} - \bar{z}_i\partial_{\bar{z}_i} + \frac{1}{4}\bar{z}_i z_i) + 2\eta \sum_{\substack{i=1 \\ i\neq j}}^N \frac{1}{(z_i - z_j)}(\partial_{\bar{z}_i} - \frac{1}{4}z_i) \quad , \quad (60)$$

with the Laughlin wavefunction as the ground-state,

$$\psi_0 = \prod_{i<j}(z_i - z_j)^\eta \ \exp\{-\frac{1}{4}\sum_i \bar{z}_i z_i\} \quad .$$

The second one can be viewed as a special case of (52):

$$\begin{aligned} H = \quad & \frac{1}{2}\sum_{i=1}^N(-4\partial_{\bar{z}_i}\partial_{z_i} + z_i\partial_{z_i} - \bar{z}_i\partial_{\bar{z}_i} + \frac{1}{4}\bar{z}_i z_i) + 2\eta \sum_{\substack{i=1 \\ i\neq j}}^N \frac{1}{(z_i - z_j)}(\partial_{\bar{z}_i} - \frac{1}{4}z_i) \\ & + 2\eta^2 \sum_{\substack{i,j,k=1 \\ i\neq j; i\neq k}}^N \frac{1}{(z_i - z_j)(\bar{z}_i - \bar{z}_k)} \quad , \end{aligned} \quad (61)$$

with the complex conjugate of the Laughlin wavefunction as the ground-state,

$$\psi_0 = \prod_{i<j}(\bar{z}_i - \bar{z}_j)^\eta \ \exp\{-\frac{1}{4}\sum_i \bar{z}_i z_i\} \quad .$$

In parallel to the earlier treatment, the above two Hamiltonians can be mapped to a set of N decoupled harmonic oscillators on a plane. One can also show the existence of a *linear* $W_{1+\infty}$ algebra for these models.

5. CONCLUSION

In conclusion, we have studied the degeneracy structure of the anisotropic oscillator with rational frequency ratio, both with and without singular terms, by mapping the given anisotropic oscillator to a regular isotropic harmonic oscillator with equal frequencies. This correspondence enabled one to easily write down the $SU(N)$ symmetry algebra for the N-dimensional anisotropic oscillator. This was done in two different ways and the $SU(N)$ generators are free from the ambiguities encountered earlier.

A similar technique was also applied to a recently studied four-particle model in one dimension, with both pair-wise and four-body inverse-square interactions and it was shown that the underlying algebraic structure of this model is nothing but that of the free harmonic oscillators. We explicitly proved the quantum integrability of this model and also computed some eigenfunctions in the Cartesian basis. A method to construct new solvable models was also outlined. It will be of great interest to study these one and higher dimensional models explicitly. We further studied two-dimensional models having Laughlin type wavefunctions as ground-state and establish its connection with free oscillators. This enabled us to realize the Laughlin wavefunctions as the highest weight vectors of a linear $W_{1+\infty}$ algebra. This analysis also needs further study since some of these models can be used to implement the Jain picture of the fractional quantum Hall effect. We hope to come back to some of these points in future work.

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